Perturbation Methods for Satellite Orbits

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The literature in astrodynamics abounds with perturbation techniques for satellite orbits. Various formulations have been generated in terms of orbit elements, the satellite position and velocity vectors, or combinations thereof. The computational effectiveness of any perturbation scheme depends largely on the definitions used for the dynamic state variables. Some methods are aimed at long-range predictions and orbit lifetime studies, others at short-range predictions for guidance. This paper may serve as an introduction to this field for the nonspecialist, in that it reviews the classical variation-of-parameters technique and discusses several engineering analyses that were generated in the post-Sputnik era. It also points to some connections between these relatively simple approaches and more elaborate methods of celestial mechanics. Thus it may contribute toward a comparison of several "professional" approaches whose relative merits are often debated among experts.

I. INTRODUCTION

This paper is a discussion of various perturbation techniques for satellite orbits which were investigated by the author and his colleagues during the past few years. The effort began with a tutorial "orbit seminar" several years ago and it seemed appropriate to collect some of this material here as a companion paper for R. B. Blackman's "Methods of Orbit Refinement."

It is a symptom of our times that aerospace engineers are taking a new look at the established methods of dynamical astronomy. The orbital geometries and vehicle characteristics encountered with artificial celestial bodies often require departures from the formulations of classical astronomy and, in fact, have stimulated several new (or at least independent) approaches during the post-Sputnik era. The number of publications in this time has been formidable, and in many discussions the names attached to various formulations serve as passwords for the ideas they represent. The uninitiated find themselves at a loss

concerning the methods that stand behind these names, their degree of originality, and their relations with each other.

In view of this situation the following article is addressed to two kinds of readers:

(*i*) The newcomers in the field of orbital mechanics who seek a tutorial survey and an introduction to some of the literature. A bare minimum of definitions is given for their benefit; a discussion of basic order-of-magnitude relations and certain intuitive notions which would strengthen the beginner's grasp of the physical problem had to be omitted for lack of space but can be found in the literature.[1,2]

(*ii*) The specialists in orbital mechanics who have not had occasion to correlate some of the better-known contributions in the literature and who may find this work a step in that direction. Typical issues in such comparisons are the choice of coordinates, the accuracy and elegance achieved by various transformations of the variables, and the precision obtainable from series expansions of the solution in terms of various small parameters.

The simultaneous need for conciseness of presentation and discussion of certain analytic detail presents somewhat of a dilemma. As a compromise, much of the development between the explicitly quoted results is covered in a descriptive way and the reader is referred to the literature for all standard derivations.[1-4] Most of our discussion concerns orbits of moderate eccentricity, which are representative of satellite missions. However, in many places an extension to the highly eccentric orbits of space probes follows readily.

We begin by devoting Section II to a statement of the fundamental equations of motion, the definition of so-called orbit parameters, and a description of various disturbing functions. Section III summarizes the classical treatment of satellite perturbations as gradual changes of the orbit parameters. (From a general point of view, this formulation, due to Lagrange, is derivable from the canonical systems governing the satellite problem.) It is hoped that this covers a sufficient amount of standard material to introduce the concepts and the parlance of orbital mechanics.

In Section IV we examine several perturbation methods for aerospace applications which are based on variously defined spherical and moving Cartesian coordinates. This includes the well-known contributions by Blitzer et al., Anthony et al., and Roberson. They could serve as an introduction to the discussion of more elaborate formulations by King-Hele et al. and Brenner et al. In Section V we treat one more formulation in this general category which was specifically designed for guidance studies.

A logical continuation of this paper would cover the methods of Breakwell et al. and Diliberto, Kyner's averaging technique, and the one suggested by Struble. Ultimately the hierarchy of perturbation methods leads to the Hamilton-Jacobi techniques expounded by Brouwer, Garfinkel, and Vinti. These represent a very popular approach to higher-order perturbations and the coupling between simultaneous disturbances of satellite orbits.

II. PRELIMINARIES AND DEFINITIONS

We remember that the underlying phenomenon of undisturbed satellite motion (in a central force field, i.e. around a spherically symmetric body) is Newton's law of inverse square attraction. In a Cartesian coordinate system this spells out to be

$$\ddot{x} = -(Gm_e x/r^3) \tag{1}$$

$$\ddot{y} = -(Gm_e y/r^3) \tag{2}$$

$$\ddot{z} = -(Gm_e z/r^3) \tag{3}$$

where G is the universal gravitational constant, m_e the central mass, and we shall usually take $Gm_e = k$ for brevity. m_e shall be the mass of the earth in all our discussions. [Strictly speaking, formulas (1) to (3) should show the sum of m_e and the satellite mass instead of just m_e.] r is the distance from the origin, and dots indicate time derivatives. The x-y plane is usually taken to coincide with the equator, while the positive x axis points to the vernal equinox. The above equations simply state that each acceleration component is due to the corresponding component of the gravitational attraction — the minus signs indicating a direction toward the origin. The solutions of (1) to (3) are the well-known Kepler orbits — ellipses, parabolas, and hyperbolas.

Such orbits can be conveniently described by a set of six parameters that give the plane of the motion, the shape of the orbit and its orientation in that plane, and the timing of the satellite motion along this path. These quantities may be considered the constants of integration for a solution of (1) to (3). A standard set of such orbit elements for elliptic motion is illustrated in Fig. 1. They are:

a, the semimajor axis,
e, the eccentricity,
ω, the argument of perigee, (4)
i, the inclination,
Ω, the nodal angle, and
τ, the time of perigee passage.

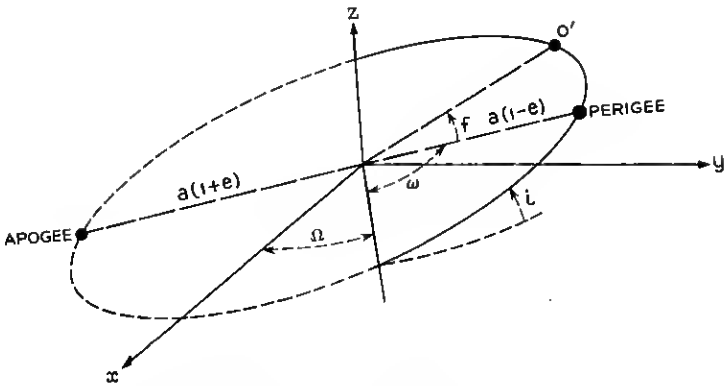

Fig. 1 — Standard set of orbit elements for elliptic motion.

The last quantity establishes a time scale for the entire motion in that it serves as "epoch" and fixes one particular passage through perigee. If the satellite has swept out the angle f since that passage, the elapsed time is given by

$$t - \tau = (a^3/k)^{\frac{1}{2}} \left\{ 2 \tan^{-1} \left[\left(\frac{1-e}{1+e} \right)^{\frac{1}{2}} \tan \frac{f}{2} \right] - e\,(1 - e^2)^{\frac{1}{2}} \frac{\sin f}{1 + e \cos f} \right\}_0^f \tag{5}$$

where t is the time pertaining to the position 0′. f is known as the true anomaly and (5) holds for all values of this angle. If we set $f = 2\pi$ this corresponds of course to a full revolution around the orbit, and the elapsed time interval is

$$T = 2\pi\,(a^3/k)^{\frac{1}{2}}, \tag{6}$$

which is known as the anomalistic period. The instantaneous position 0′ can also be defined in terms of other angles, the so-called eccentric anomaly E or the mean anomaly M, which will be defined later. They can be related to time in similar ways.

The parameters in (4) represent a typical set of orbit elements. The position and velocity vectors at some epoch are an alternative suggested by (1)–(3). Most astrodynamical theories use variations and combinations of all these, but from the general standpoint of analytical dynamics most sets of six parameters (if they are independent of each other) may be regarded as sets of canonic variables. Before we proceed to detailed formulations we examine briefly the various physical disturbances which cause these parameters to change in time.

2.1 *The Effect of Extraterrestrial Gravitation*

If we consider the attractions from masses other than the earth we speak of "extraterrestrial" gravitation. In the presence of a disturbing body P, (1) becomes

$$\ddot{x} = -\frac{Gm_e x}{r^3} - Gm_p\left(\frac{x - x_p}{r_{ip}^3} + \frac{x_p}{r_p^3}\right) \tag{7}$$

where

$$m_p = \text{the mass of } P,$$

$$r_{ip} = [(x - x_p)^2 + (y - y_p)^2 + (z - z_p)^2]^{\frac{1}{2}},$$

the distance from the satellite to P, and

$$r_p = [x_p^2 + y_p^2 + z_p^2]^{\frac{1}{2}},$$

the geocentric distance of P.

The corresponding equations for $\ddot{y}$ and $\ddot{z}$ are obvious. Now it is often convenient in analytical dynamics to express the disturbing terms in $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ as partial derivatives $(\partial\tilde{R}/\partial x)$, $(\partial\tilde{R}/\partial y)$, $(\partial\tilde{R}/\partial z)$ of a disturbing function $\tilde{R}$. For the present case we would have

$$\tilde{R} = Gm_p\left[\frac{1}{r_{ip}} - \frac{xx_p + yy_p + zz_p}{r_p^3}\right], \tag{8}$$

as can be easily verified.

We see that the ratio of the second term in (7) to the first is of the order $m_p r^3/m_e r_p^3 = \kappa$. Typical values for κ in the planetary system are equal to or less than 10^{-5}. Its smallness is vital to the entire rationale of a perturbation technique.

2.2 *The Effect of the Earth's Oblateness*

The potential field for the nonspherical earth can be represented to various levels of accuracy by a series of spherical harmonics. If we restrict ourselves to terms with rotational symmetry about the polar axis, we obtain the following disturbing function

$$\tilde{R} = \frac{Gm_e R^2}{r^3}\left[\frac{J}{3}(1 - 3\sin^2\varphi) + \frac{H}{5}\frac{R}{r}(3\sin\varphi - 5\sin^3\varphi) + \cdots\right] \tag{9}$$

where

R = the earth's equatorial radius,
φ = the geocentric latitude of the satellite,
$J = 1.6239 \times 10^{-3}$, and
$H = 6.04 \times 10^{-6}$.

This two-term series is sufficiently accurate for our purposes.

2.3 *The Effect of Atmospheric Drag*

The resistance encountered by a satellite from the atmosphere is a subject of considerable uncertainty and continued research. For one thing, the density of atmospheric gases as a function of geographic location, altitude and time is not well known; moreover, the laws of interaction between a satellite and this rarefied medium are incompletely understood. Doubts exist as to the transition from a continuum behavior of the atmosphere to the gas-kinetic regime and the extent to which electric interactions play a role. Nevertheless, the classical drag law yields useful results in many cases and we shall concentrate on it. We let

$$F_D = -(C_D A/2)\rho v_a^2 \tag{10}$$

where

F_D = the total drag force on the satellite
A = the frontal area of the satellite
C_D = the drag coefficient
ρ = the atmospheric density
v_a = the satellite velocity relative to the atmosphere.

The monotonic decay of ρ with altitude covers approximately ten orders of magnitude within typical satellite altitudes and remains the subject of extensive study. The relative velocity v_a is simply the difference between the satellite's inertial velocity vector $\mathbf{v}(\dot{x},\dot{y},\dot{z})$ and the rotational velocity of the atmosphere $V = r\sigma \cos \varphi$, where σ can usually be taken as the earth's angular motion (the diurnal rate) and V always points due east. One is frequently justified in employing an approximate vector representation of (10):

$$\mathbf{F}_D \approx (C_D A/2)\ \rho\ v(\mathbf{V} - \mathbf{v}). \tag{11}$$

For typical earth satellites this force is at least two orders of magnitude smaller than the central attraction, i.e., $\kappa \leqq 10^{-2}$.

2.4 *The Effect of Radiation Pressure*

As the reader knows, solar illumination exerts some pressure on every satellite. The magnitude of this force depends on the reflectivity and geometry of the satellite and, strictly speaking, on the distance from the satellite to the sun. It frequently suffices to represent this disturbance as a constant force β per unit mass and to note that it is many orders of magnitude smaller than the central gravity force.

III. PERTURBATIONS IN THE ELEMENTS

The six orbit elements [see (4)] were constants for the case of central inverse-square attraction. However, if any additional forces act on the satellite these parameters will be subject to change. To emphasize their time dependence we might write them as $a(t)$, $e(t)$ etc. In fact, their numerical values at any time t describe the ellipse the orbiting body would follow if all perturbations vanished as of that instant. This trajectory is obviously tangent to the actual flight path at t and is known as the "osculating" orbit. The relation between the satellite position in the osculating orbit and in the x, y, z frame follows from the geometry of conic section trajectories:

$$
\begin{aligned}
x &= \frac{a(1-e^2)}{1+e\cos f}\,[l_1 \cos f + l_2 \sin f] \\
y &= \frac{a(1-e^2)}{1+e\cos f}\,[m_1 \cos f + m_2 \sin f] \\
z &= \frac{a(1-e^2)}{1+e\cos f}\,[n_1 \cos f + n_2 \sin f]
\end{aligned}
\tag{12}
$$

where $l_1, l_2, \cdots n_2$ are functions of i, ω, and Ω. Hence x, y, z are representable as functions of a, e, i, ω, Ω, f. [As mentioned with (5), we could also work in terms of the independent variable E or M instead of f.] However, the complete definition of the osculating orbit also entails that

$$
\begin{aligned}
\dot{x} &= \frac{\partial x}{\partial f}\,(a,e,i,\omega,\Omega,f)\,\frac{df}{dt} \\
\dot{y} &= \frac{\partial y}{\partial f}\,(a,e,i,\omega,\Omega,f)\,\frac{df}{dt} \\
\dot{z} &= \frac{\partial z}{\partial f}\,(a,e,i,\omega,\Omega,f)\,\frac{df}{dt}
\end{aligned}
\tag{13}
$$

where a, e, i, ω, Ω, τ are treated as constants. In other words, the velocity as well as the position in the osculating orbit are representative of the actual motion. This is the full extent of the "condition of osculation."

A large part of classical celestial mechanics has been based on the concept of osculating orbits, and during the post-Sputnik era Lagrange's classical treatment of the perturbations in these elements has been exploited *ad ultimo*. Its inclusion in this article is justified mainly by the need for completeness in an introductory survey such as this. It also serves as a point of reference for the nonclassical "perturbations in the coordinates" in the next section and for the Hamilton-Jacobi techniques frequently used by astronomers.

In essence, the Lagrange method consists of transforming the basic equations

$$
\begin{aligned}
\ddot{x} &= -\frac{Gm_e x}{r^3} + \frac{\partial \tilde{R}}{\partial x} \\
\ddot{y} &= -\frac{Gm_e y}{r^2} + \frac{\partial \tilde{R}}{\partial y} \\
\ddot{z} &= -\frac{Gm_e z}{r^3} + \frac{\partial \tilde{R}}{\partial z}
\end{aligned}
\tag{14}
$$

to six first-order equations in the orbit elements and approximating their solutions by quadratures. Remembering that these parameters represented the constants of integration for the Kepler problem, (1)–(3), we note that the transition to $a(t)$, $e(t)$, $\cdots$ is nothing but Lagrange's "variation of constants" designed to accommodate the terms $[\partial\tilde{R}/\partial(x,y,z)]$ in (14). In the process of transforming (14) by means of (12) we avoid the occurrence of second derivatives of the orbit elements by demanding that

$$\frac{dx}{dt} = \frac{\partial x}{\partial t}, \text{ i.e.,}$$

$$\frac{\partial x}{\partial a}\dot{a} + \frac{\partial x}{\partial e}\dot{e} + \frac{\partial x}{\partial i}\frac{\partial i}{\partial t} + \frac{\partial x}{\partial \omega}\dot{\omega} + \frac{\partial x}{\partial \Omega}\dot{\Omega} + \frac{\partial x}{\partial \tau}\dot{\tau} = 0 \tag{15}$$

etc. This of course results in the reduction of the system of three second-order equations (14) to six first-order equations. Equations (15) are simply a restatement of (13), the condition of osculation.

In principle, (14) could be transformed by a straightforward substitution of (12). In order to simplify the algebra, however, one may

symmetrize these equations to a form where all the labor reduces to the evaluation of such quantities as

$$\left(\frac{\partial x}{\partial \alpha_i}\frac{\partial \dot{x}}{\partial \alpha_j} - \frac{\partial x}{\partial \alpha_j}\frac{\partial \dot{x}}{\partial \alpha_i}\right) + \left(\frac{\partial y}{\partial \alpha_i}\frac{\partial \dot{y}}{\partial \alpha_j} - \frac{\partial y}{\partial \alpha_j}\frac{\partial \dot{y}}{\partial \alpha_i}\right) + \left(\frac{\partial z}{\partial \alpha_i}\frac{\partial \dot{z}}{\partial \alpha_j} - \frac{\partial z}{\partial x_j}\frac{\partial \dot{z}}{\partial \alpha_i}\right) = [\alpha_i, \alpha_j], \tag{16}$$

with $i, j = 1 \cdots 6$.

The shorthand symbol that we have adopted for this expression is known as a Lagrange bracket, and α_i and α_j stand for any two of the orbit elements. These brackets have the properties

$$[\alpha_i, \alpha_i] = 0, \qquad [\alpha_i, \alpha_j] = -[\alpha_j, \alpha_i] \tag{17}$$

and

$$\frac{d}{dt}[a_i, a_j] = 0,$$

which make them useful devices in numerous manipulations of analytic dynamics. With their help the equations (14) become

$$\dot{a} = -\frac{2a^2}{k}\frac{\partial \tilde{R}}{\partial \tau} \tag{18}$$

$$\dot{e} = -\frac{a(1-e^2)}{ke}\frac{\partial \tilde{R}}{\partial \tau} - \frac{1}{e}\left(\frac{1-e^2}{ka}\right)^{\frac{1}{2}}\frac{\partial \tilde{R}}{\partial \omega} \tag{19}$$

$$\frac{di}{dt} = \frac{1}{[ka(1-e^2)]^{\frac{1}{2}}\sin i}\left[\cos i\,\frac{\partial \tilde{R}}{\partial \omega} - \frac{\partial \tilde{R}}{\partial \Omega}\right] \tag{20}$$

$$\dot{\Omega} = \frac{1}{[ka(1-e^2)]^{\frac{1}{2}}\sin i}\frac{\partial \tilde{R}}{\partial i} \tag{21}$$

$$\dot{\omega} = \left(\frac{1-e^2}{ka}\right)^{\frac{1}{2}}\frac{1}{e}\left[\frac{\partial \tilde{R}}{\partial e} - \frac{e\cot i}{1-e^2}\frac{\partial \tilde{R}}{\partial i}\right] \tag{22}$$

and a corresponding equation for $\dot{\tau}$ which will be discussed a little later. The five equations given here describe the changing geometry of the satellite orbit. All six equations together are known as Lagrange's planetary or "variational" equations.

In (18)–(22) we assumed that the perturbing forces were conservative, i.e., expressible as $(\partial\tilde{R}/\partial x)$, $(\partial\tilde{R}/\partial y)$, and $(\partial\tilde{R}/\partial z)$. In some situations, as for example in the case of drag, this is not so. Under these conditions it is convenient to represent the disturbing force by com-

ponents S, T, N which are in the radial direction, the direction of increasing true anomaly, and normal to the orbit plane, respectively. The derivation leading to (18)–(22) can be repeated with the appropriate modifications to yield a set of differential equations with S, T, N in the right-hand sides. For example

$$\dot{a} = 2\left[\frac{a^3}{k(1-e^2)}\right]^{\frac{1}{2}} [S\ e \sin f + T\ (1 + e \cos f)], \text{ etc.} \tag{23}$$

These are known as Gauss's form of the planetary equations. We note that they contain the true anomaly as an independent variable. More will be said about this presently.

It is possible to show that the planetary equations, especially in the last form, lend themselves to an alternative derivation which appeals to intuition. It is based on the idea that any continuously acting perturbation may be interpreted as a sequence of infinitesimal impulses whose cumulative time response can be represented by a convolution integral. This approach leads to equations like (18)–(23) without the manipulations involving Lagrange brackets.[6]

Inspection of (18)–(22) shows that their nonlinear right-hand sides preclude an exact solution except for very special forms of $\tilde{R}$. Unfortunately, none of the perturbations encountered in nature fall into this category. One therefore resorts to a process of successive approximations.

Assuming that all disturbances represented by $\tilde{R}$ are small in relation to the central attraction (i.e.,

$$\left(\frac{\partial \tilde{R}}{\partial(x,y,z)}\right) \Big/ \frac{k(x,y,z)}{r^{(0)3}} = \kappa \ll 1,$$

as discussed in Section II) we consider the solution for the undisturbed motion, i.e. the Kepler problem, as a "zero-order" approximation to the actual case. Let its parameters be denoted $a^{(0)}$, $e^{(0)}$ $\cdots$. If we insert them into the right-hand sides of (18)–(22), these equations reduce to quadratures yielding a first approximation to the effects of $\tilde{R}$ on the orbit. These results are denoted $a^{(1)}$, $e^{(1)}$ $\cdots$ and known as "first-order" perturbations. We observe that $(a^{(1)}/a^{(0)})$, $(e^{(1)}/e^{(0)})$, $\cdots = 0(\kappa)$. In principle, this process can be repeated indefinitely by substituting $a^{(n-1)}$ $\cdots$ into the right-hand sides to integrate for $a^{(n)}$ $\cdots$. The limit is usually reached when the results for a, e $\cdots$ have settled or human endurance is exhausted. (The latter constraint may be eventually eliminated by computer routines for symbol manipulations.) The convergence of this

process to the exact solution has been established by Poincaré and is of fundamental interest to the mathematician. Suffice it here to say that the "smallness" of perturbations discussed in Section II should be such as to justify the iterative process.

When the right-hand sides of (18)–(22) are written out explicitly for any particular case, they tend to become awkward because a transcendental angle-time relation like (5) enters. Since the geometric description of a perturbation $\bar{R}$ (or S,T,N) usually involves an angle like one of the anomalies very directly, it is convenient to use one of them as independent variable. The time relation which interconnects the anomalies for an osculating orbit (Kepler's equation) can be stated in terms of the eccentric anomaly E, the true anomaly, f and the mean anomaly M as follows:

$$\begin{aligned} E - e\sin E &= 2\tan^{-1}\left[\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}\tan\frac{f}{2}\right] - \frac{e(1-e^2)^{\frac{1}{2}}\sin f}{1+e\cos f} \\ &= (k/a^3)^{\frac{1}{2}}(t-\tau) = M. \end{aligned} \tag{24}$$

This may serve as a definition of E and M. The quantity $(k/a^3)^{\frac{1}{2}} \equiv n$ is referred to as the "mean angular rate."

If we work in terms of the true anomaly, we can write the left-hand sides of Lagrange's equations as

$$\dot{a} = \frac{da}{df}\frac{df}{dt}, \text{ etc.}$$

where an expression must now be found for $\dot{f}$. If we choose to consider f as the true anomaly in some osculating orbit valid at time t_0, then it can be related to time by (24) in terms of the unperturbed elements $a^{(0)} \equiv a_0$, $e^{(0)} \equiv e_0$ etc. We call it an "unperturbed" anomaly and designate it by $f^{(0)}$. From (24) one finds

$$\dot{f}^{(0)} = \left[\frac{k}{a_0^3(1-e_0^2)^3}\right]^{\frac{1}{2}}(1+e_0\cos f^{(0)})^2 \tag{25}$$

which transforms (18)–(22) to

$$\frac{da^{(1)}}{df^{(0)}} = -2\left[\frac{a_0^7(1-e_0^2)^3}{k^3}\right]^{\frac{1}{2}}(1+e_0\cos f^{(0)})^{-2}\frac{\partial\bar{R}}{\partial\tau_0}, \text{ etc.} \tag{26}$$

When integrated, these expressions represent first-order perturbations in terms of the unperturbed anomaly, i.e., $a^{(1)}(f^{(0)})$, etc.

They suggest the following procedure for generating a first-order satellite ephemeris: if t_0 is the starting epoch we evaluate $a^{(1)}(f^{(0)})$ etc.

between the limits $f_0^{(0)}$ and $f_1^{(0)}$. The time t_1 at the upper limit follows from (24) in terms of a_0, e_0 $\cdots$. Using t_1 and $\tilde{a}_1 = a_0 + a^{(1)}(f_1^{(0)})$, etc., in (24), we find $\bar{f}_1$, the true anomaly for the new osculating orbit. Changing the notation from $\tilde{a}_1$ to a_1, etc., and $\bar{f}_1$ to $f_1^{(0)}$, we can now repeat the procedure for the next integration interval. The "updating" of orbit elements in the right-hand side of (26) amounts to a partial allowance for higher-order perturbations, while the recalculation of f at the beginning of each step represents essentially a first-order perturbation of the true anomaly.

Instead of doing the latter by discrete increments, we can work with a "perturbed" anomaly by differentiating (24) with proper allowance for the time dependence of a, e and τ. Using (25) one finds that

$$\begin{aligned} \frac{df}{df^{(0)}} = \frac{\dot{f}}{\dot{f}^{(0)}} = 1 &- \frac{2(1-e^2)^{\frac{1}{2}}a^2}{k(1+e\cos f)^2}\left(\frac{\partial\tilde{R}}{\partial a}\right) - \frac{a(1-e^2)^{\frac{3}{2}}}{ek(1+e\cos f)^2}\frac{\partial\tilde{R}}{\partial e} \\ &+ \frac{a^{\frac{1}{2}}(1-e^2)}{k^{\frac{1}{2}}(1+e\cos f)^3}\left[(1+\cos f)\tan\frac{f}{2}\right. \\ &\left.- (2e^2-1)\sin f - \frac{(1-e^2)e\sin f\cos f}{1+e\cos f}\right] \\ &\times\left[\frac{a(1-e^2)}{ke}\frac{\partial\tilde{R}}{\partial\tau} + \frac{1}{e}\left(\frac{1-e^2}{ak}\right)^{\frac{1}{2}}\frac{\partial\tilde{R}}{\partial\omega}\right] \end{aligned} \tag{27}$$

where $(\partial\tilde{R}/\partial a)$ means that $\tilde{R}$ is to be differentiated with respect to the semimajor axis wherever the latter appears explicitly but not when it is contained in n. This avoids the occurrence of a term with $(t - \tau)$. An expression analogous to (27) can be derived in terms of S,T,N; see for example Ref. 7, p. 4. Now, it is immaterial in a first-order approximation such as (27) whether we consider f or $f^{(0)}$ as the independent variable in the right-hand side. Let us assume the former and use the symbol $df/df^{(0)} = \nu(f)$. Then (26) becomes

$$\frac{da^{(1)}}{df} = \frac{-2}{\nu}\left[\frac{a_0^{\,7}(1-e_0^{\,2})^3}{k^3}\right]^{\frac{1}{2}}(1+e_0\cos f)^{-2}\frac{\partial\tilde{R}}{\partial\tau_0}, \text{ etc.} \tag{28}$$

The integration procedure now runs between consecutive limits f_0, f_1, $\cdots$ f_j, with updating being required only in the orbit elements. The corresponding epochs t_j are of course computable by substituting a_j, e_j $\cdots$ f_j into (24). The relative advantages of integrating the perturbative equations in terms of $f^{(0)}$ or f depend on the problem at hand. As we shall see later, the choice between an unperturbed or perturbed independent variable is available in most perturbation methods.

Up to this point we have restricted our discussion to the first five

orbit parameters, which describe the geometry of the osculating orbit. Wherever τ appeared, as in (24), we assumed that it would be available from a suitable sixth equation. This parameter is needed to correlate the independent variable, such as f, with time.

An equation for τ, corresponding to (18)–(22), can be obtained by the process outlined before, which yields

$$\tau = \frac{2a^2}{k} \frac{\partial \tilde{R}}{\partial a} + \frac{a(1 - e^2)}{ke} \frac{\partial \tilde{R}}{\partial e}. \tag{29}$$

Sometimes it is convenient to work with the slightly different parameter $\chi = -n\tau$. The differential equation for it reads

$$\dot{\chi} = -2(a/k)^{\frac{1}{2}} \frac{\partial \tilde{R}}{\partial a} - \frac{1 - e^2}{e(ak)^{\frac{1}{2}}} \frac{\partial \tilde{R}}{\partial e}. \tag{30}$$

[A superficial comparison of these two equations gives the startling impression that (30) is obtained by multiplying (29) with $-n$, thus neglecting the $\dot{n}$ term that should appear. This term is really absorbed in the difference between $(\partial \tilde{R}/\partial a)_\tau$ and $(\partial \tilde{R}/\partial a)_\chi$, as implied by the two equations; i.e. partial derivatives of $\tilde{R}$ with respect to the semimajor axis, holding τ or χ constant as required.]

One could transform (29) and (30) to f (or E) as independent variable and pursue the quadrature as we did before. However, since $\partial \tilde{R}/\partial a$ involves $\partial f/\partial a$ (or $\partial E/\partial a$), we notice from (24) that this introduces the factors $\tan^{-1} \{[(1 - e)/(1 + e)]^{\frac{1}{2}} \tan (f/2)\}$ (or E) into the integrands for $\tau^{(1)}$ (or $\chi^{(1)}$). They can be quite awkward.

Several devices have been developed to circumvent this difficulty. According to one approach we transform (29) from τ to $M = n(t - \tau)$. The necessary compensating factors arise thereby which eliminate all aperiodic terms. Transforming the integrated equation back to τ, we have

$$\begin{aligned} \tilde{\tau}_1 \tilde{n}_1 = \tau_0 n_0 + t_1(\tilde{n}_1 - n_0) + \left[\frac{a_0{}^3(1 - e_0{}^2)^3}{k}\right]^{\frac{1}{2}} \int_{f_0{}^{(0)}}^{f_1{}^{(0)}} \\ (1 + e_0 \cos f^{(0)})^{-2} \left\{ -\frac{3k^{\frac{1}{2}}}{2a_0{}^{\frac{5}{2}}} a_1{}^{(1)} + \frac{1 - e_0{}^2}{e_0(ka_0)^{\frac{1}{2}}} \frac{\partial \tilde{R}}{\partial e_0} \right. \\ \left. + 2(a_0/k)^{\frac{1}{2}} \left(\frac{\partial \tilde{R}}{\partial a_0}\right) \right\} df^{(0)} \end{aligned} \tag{31}$$

where

$$\tilde{n}_1 = (k/\tilde{a}_1{}^3)^{\frac{1}{2}}, \qquad n_0 = (k/a_0{}^3)^{\frac{1}{2}}, \qquad \tilde{\tau}_1 = \tau_0 + \tau_1{}^{(1)}, \qquad \tilde{a}_1 = a_0 + a_1{}^{(1)},$$

and $(\partial \tilde{R}/\partial a_0)$ has the previously established meaning. Equation (31) can

be obtained in terms of the perturbed anomaly f if it is understood that $a_1^{(1)}$ in the quadrature and in $\tilde{a}_1$ is obtained by (28) and if the integrand of (31) is multiplied by $1/\nu$.

3.1 *Oblateness Effects*

We briefly illustrate some results from Lagrange's method. In the well-known example of oblateness perturbations, the first-order solutions for a, e, and i turn out to be entirely periodic and not very interesting.[8] The remaining elements, however, exhibit secular terms. Using only the J-term of (9) we find from (21), (22) and (31)

$$\bar{\Omega}_1 = \Omega_0 - \frac{JR^2}{a_0^2(1 - e_0^2)^2} (\cos i_0)(f_1 - f_0) + \text{periodic terms} \tag{32}$$

$$\begin{aligned} \bar{\omega}_1 = \omega_0 - (\cos i_0)(\bar{\Omega}_1 - \Omega_0) + \frac{JR^2}{a_0^2(1 - e_0^2)^2}\left(1 - \frac{3}{2}\sin^2 i_0\right) \\ \cdot (f_1 - f_2) + \frac{JR^2}{a_0^2 e_0(1 - e_0^2)^2} \times \text{p.t.} \end{aligned} \tag{33}$$

$$\begin{aligned} \bar{\tau}_1 = \tau_0 \frac{n_0}{\bar{n}_1} - \frac{n_0}{\bar{n}_1} \frac{JR^2}{a_0^2(1 - e_0^2)^3} \\ \times \{t(1 + e_0 \cos f)^3 [1 - 3 \sin^2 i_0 \sin^2 (\omega_0 + f)]\}_{f_0}^{f_1} \\ + \frac{1}{\bar{n}_1}(1 - e_0^2)^{\frac{1}{2}} [(\bar{\omega}_1 - \omega_0) + (\cos i_0)(\bar{\Omega}_1 - \Omega_0)] \\ - \frac{JR^2}{\bar{n}_1 a_0^2(1 - e_0^2)^{\frac{3}{2}}}\left(1 - \frac{3}{2}\sin^2 i_0\right)(f_1 - f_0) + \text{p.t.} \end{aligned} \tag{34}$$

where f, f_1, f_0 represent the unperturbed anomaly. (We have omitted the superscript zero for convenience.) Equation (32) confirms the well-known secular behavior of the node. It turns out to shift westward for $0 < i_0 < \pi/2$ and eastward for $\pi/2 < i_0 < \pi$. At $i_0 = \pi/2$ it remains stationary, as would be expected from symmetry.

The secular component of $\omega^{(1)}$, according to (33), reduces to the well-known term

$$\frac{JR^2(5 \cos^2 i_0 - 1)}{2a_0^2(1 - e_0^2)^2}.$$

It represents an advance of perigee for $0 \leqq i_0 < 63°26'$ and for $116°34' < i_0 \leqq \pi$. For $63°26' < i_0 < 116°34'$ perigee regresses, and at the "critical" angles $63°26'$ and $116°34'$ it is reduced to periodic motions (as far as the first-order analysis indicates). We note that the periodic terms in

(33) contain e_0 in the denominator, and we expect $\omega^{(1)}$ to behave unstably for near-circular orbits (as one might expect for geometric reasons). Indeed, this singular behavior can be expected also in other examples, according to (19), (22) and (31). Furthermore, some difficulties will arise with small values of i_0, according to (20) and (21). These cases of near-circular and near-equatorial orbits can be accommodated by redefining the orbit elements in various ways. While such modified elements are less accessible to a geometric interpretation, they do not encumber the calculation of perturbed satellite positions as a function of time. For the sake of brevity we must forego additional details here.

3.2 *Luni-Solar Gravitation*

We omit a discussion of $\tilde{a}_1$, since it shows periodic perturbations only. Substitution of (8) into (19)–(22) and (31) yields

$$\tilde{e}_1 = e_0 - \frac{15 m_p a_0^3 e_0 h_1 h_2}{m_e r_p^5} (1 - e_0^2)^{\frac{1}{2}} \left\{ \tan^{-1}\left[\left(\frac{1 - e_0}{1 + e_0}\right)^{\frac{1}{2}} \tan\frac{f}{2}\right)\right]\right\}_{f_0}^{f_1} + \text{p.t.} \tag{35}$$

where

$$h_1 = l_1 x_p + m_1 y_p + n_1 z_p$$

$$h_2 = l_2 x_2 + m_2 y_p + n_2 z_p$$

$$\tilde{i}_i = i_0 + \frac{3 m_p a_0^3}{m_e r_p^5 \sqrt{1 - e_0^2}} [(1 + 4e_0^2) h_1 h_{2_i} - (1 - e_0^2) h_2 h_{1_i}] \cdot \left\{ \tan^{-1}\left[\left(\frac{1 - e_0}{1 + e_0}\right)^{\frac{1}{2}} \tan\frac{f}{2}\right]\right\}_{f_0}^{f_1} + \text{p.t.} \tag{36}$$

$$\tilde{\Omega}_1 = \Omega_0 + \frac{3 m_p a_0^3 [(1 + 4e_0^2)(h_1^2)_i + (1 - e_0^2)(h_2^2)_i]}{2 m_e r_p^5 \sin i_0 (1 - e_0^2)^{\frac{1}{2}}} \cdot \left\{ \tan^{-1}\left[\left(\frac{1 - e_0}{1 + e_0}\right)^{\frac{1}{2}} \tan\frac{f}{2}\right]\right\}_{f_0}^{f_1} + \text{p.t.} \tag{37}$$

$$\tilde{\omega}_1 = \omega_0 + \cos i_0 (\Omega_0 - \tilde{\Omega}_1) + \frac{3 m_p a_0^3}{m_e r_p^5} (1 - e_0^2)^{\frac{1}{2}} [4h_1^2 - h_2^2 - r_p^2] \cdot \left\{ \tan^{-1}\left[\left(\frac{1 - e_0}{1 + e_0}\right)^{\frac{1}{2}} \tan\frac{f}{2}\right]\right\}_{f_0}^{f_1} + \text{p.t.} \tag{38}$$

$$\tilde{\tau}_1 = \tau_0 \frac{n_0}{\tilde{n}_1} + \frac{(1 - e_0^2)^{\frac{1}{2}}}{\tilde{n}_1} [\tilde{\omega}_1 - \omega_0 + (\tilde{\Omega}_1 - \Omega_0) \cos i_0]$$
$$+ t_1 \left(1 - \frac{n_0}{\tilde{n}_1}\right) + \frac{3Gm_p(1 - e_0^2)^2(t_1 - t_0)}{2n_0 r_p^5 (1 + e_0 \cos f_0)^2}$$
$$\cdot [r_p^2 - 3(h_1 \cos f_0 + h_2 \sin f_0)^2] + \frac{7Gm_p}{2n_0^2 \tilde{n}_1 r_p^3} \tag{39}$$
$$\cdot \left[2 + 3e_0^2 - \frac{3h_1^2}{r_p^2}(1 + 4e_0^2) - \frac{3h_2^2}{r_p^2}(1 - e_0^2)\right]$$
$$\cdot \left\{\tan^{-1}\left[\left(\frac{1 - e_0}{1 + e_0}\right)^{\frac{1}{2}} \tan \frac{f}{2}\right]\right\}_{f_0}^{f_1} + \text{p.t.}$$

The subscripts i in (36) and (37) denote partial differentiation with respect to the inclination. The secular term of $e^{(1)}$ is a significant feature of our present results. It indicates that even near-circular satellite orbits can experience an unstable buildup of eccentricity due to luni-solar perturbations. The rate of this perturbation is proportional to the factor $m_p a_0^3 / m_e r_p^3$ and is usually very small; moreover, the coordinates x_p, y_p, z_p of the perturbing body are really time-dependent, which would modify the first-order result. Nevertheless, a long-period change of the eccentricity due to luni-solar gravitation has been observed in some satellite orbits.

The explicit form of the periodic terms in $a^{(1)}$ and $e^{(1)}$ contains the factor $1/e_0$, while $\Omega^{(1)}$, $\omega^{(1)}$ and $\tau^{(1)}$ contain $1/(\sin i_0)$. Again this necessitates the use of specially modified elements for low e_0 and i_0. One set of elements which is particularly suited to the problem of interplanetary perturbations is due to Strömgren. He utilized the fact the Ω, i, ω are nothing but a set of Euler angles orienting a system of orbital coordinates with one axis through pericenter, one at $f = \pi/2$, and one normal to the orbit. The rotation of these axes with respect to inertial space conveys the same information as the perturbations of Ω, i, ω. The idea is akin to Roberson's method for anticipating secular terms in the perturbation of coordinates (Section 4.3 and Ref. 15).

3.3 *Higher-Order Analyses*

The preceding examples are indicative of results to be found in the vast literature on perturbations in the osculating elements. We have merely covered the gist of this approach and several ideas which will be useful in the appraisal of other methods. Some of the better-known

contributions in terms of osculating elements are contained in papers by Krause, O'Keefe, Kozai, and Iszak.

In principle the quadratures (18)–(22) and (31) could be evaluated iteratively to generate higher-order results. This procedure rests directly on Poincaré's convergence proof, and a formal technique based on this approach is commonly attributed to Poisson. In several aerospace publications this has been done to obtain second- and third-order secular terms for oblateness effects. The algebraic labor is considerable, though typical secular terms such as (40) and (41) tend to be reasonably compact (see Refs. 7 and 9):

$$\begin{aligned}\Delta a^{(2)} &= \frac{9\pi J^2 R^4 e_0}{2a_0^3(1 - e_0^2)^5} (1 - 3 \sin^2 i_0 \sin^2 \theta_0)[1 + e_0 \sin (\omega_0 + \theta_0)]^2 \\ &\quad \times (5 \sin^2 i_0 - 4) \cos (\omega_0 + \theta_0)\end{aligned} \tag{40}$$

$$\begin{aligned}\Delta i^{(2)} &= \frac{9\pi J^2 R^4 \sin 2i_0}{4a_0^4(1 - e_0^2)^4} \{\tfrac{1}{12}e_0^2 \sin 2\omega_0 - \tfrac{1}{4}(5 \sin^2 i_0 - 4) \\ &\quad [\tfrac{1}{2}e_0^2 \sin 2\omega_0 - e_0 \cos \omega_0(\cos \theta_0 + \tfrac{1}{3} \cos 3\theta_0) \\ &\quad - e_0 \sin \omega_0(\sin \theta_0 - \tfrac{1}{3} \sin 3\theta_0)]\}.\end{aligned} \tag{41}$$

These results represent secular increments over a 2π step in θ, the central angle measured from the node, where θ_0 is the initial value of θ. Considerable emphasis must be placed in the derivation of such expressions on checks from the conservation of energy and angular momentum and duplicate execution of the algebra. (One likes to think that more elaborate explicit expressions will be attainable with the advent of computer algebra.) A notable contribution in this area was made by Merson,[5] who presents second-order secular terms for J and first-order secular terms for the next four higher harmonics of the earth's potential. He also advocates the use of "smoothed" elements which reduce the amplitude of first-order periodic terms that might otherwise be inimical to prediction accuracy.

IV. PERTURBATIONS IN THE COORDINATES

We turn now to a description of satellite motions directly in terms of the position and velocity vectors. While these are dynamically equivalent to the instantaneous orbit elements, we note that the time dependence of the dynamic state variables in this form reflects the anomalistic motion as a primary effect. Therefore the long-time, secular changes of the orbit may not be obtainable with the same clarity or pre-

cision as in terms of the osculating parameters. On the other hand, the position-time history gives a direct account of the satellite motion in space and is useful for many aerospace applications. This prospect has stimulated several engineering analyses in recent years.

To the analyst with a general background in mechanics it would seem quite natural to approach a system of equations of the type (14) by standard perturbation techniques. Thus one could assume the perturbation series

$$\begin{aligned} x(t) &= x^{(0)}(t) + \kappa x^{(1)}(t) + \kappa^2 x^{(2)}(t) + \cdots \\ y(t) &= y^{(0)}(t) + \kappa y^{(1)}(t) + \kappa^2 y^{(2)}(t) + \cdots \\ z(t) &= z^{(0)}(t) + \kappa z^{(1)}(t) + \kappa^2 z^{(2)}(t) + \cdots \end{aligned} \tag{42}$$

and determine successively higher-order terms from the appropriate governing equations, following essentially Poisson's procedure. The traditional Encke method pursues this line of attack. However, Cartesian inertial coordinates have not enjoyed as much popularity as spherical ones, which seem more compliant with the geometry of satellite orbits. In the following, therefore, we shall concentrate on reference frames of this general type.

4.1 *Perturbations in Equatorial Spherical Coordinates*

A rather well-known perturbation analysis for oblateness effects is that due to Blitzer, Weissfeld, and Wheelon.[10] It uses the conventional equatorial spherical coordinates, r, φ, ψ (see Fig. 2) in terms of which the equations of motion read:

$$\ddot{r} - r\dot{\varphi}^2 - r\cos^2\varphi\,\dot{\psi}^2 = -\frac{k}{r^2} - \frac{3JkR^2}{r^4}\left[\tfrac{1}{3} - \sin^2\varphi\right] \tag{43}$$

$$\frac{d}{dt}(r^2\dot{\varphi}) + r^2\sin\varphi\cos\varphi\,\dot{\psi}^2 = -\frac{2JkR^2}{r^3}\sin\varphi\cos\varphi \tag{44}$$

$$\frac{d}{dt}(r^2\cos^2\varphi\,\dot{\psi}) = 0. \tag{45}$$

Here we have considered the first aspherical term in the earth's potential only. Since this is a zonal harmonic and does not contain ψ, the last equation has a vanishing right-hand side. Then a first integral of this equation

$$r^2\cos^2\varphi\,\dot{\psi} = p = \text{const.}, \tag{46}$$

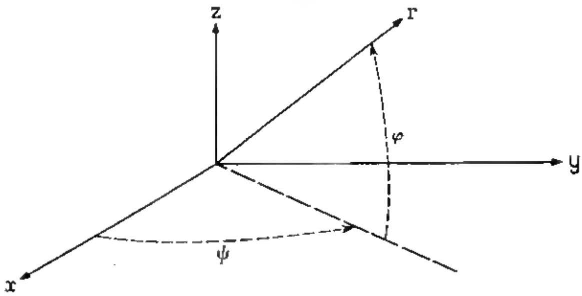

Fig. 2 — Equatorial spherical coordinate system.

representing the conservation of angular momentum about the polar axis, permits a change to ψ as independent variable. In addition, it is convenient to introduce the definitions

$$1/r = u, \qquad \tan \varphi = S, \qquad \text{and} \qquad [1/(r \cos \varphi)] = W. \tag{47}$$

Equations (43) and (44) now become

$$W'' + W = \frac{k}{p^2(1 + S^2)^{3/2}} + \frac{3JkR^2u^2}{p^2(1 + S^2)^{1/2}} \left[\frac{1}{1 + S^2} - \frac{2}{3}\right] \tag{48}$$

and

$$S'' + S = \frac{2JkR^2}{p^2} \frac{Su}{(1 + S^2)^2} \tag{49}$$

where "primes" denote differentiations with respect to ψ. In this form they are readily accessible to a perturbative procedure. We let

$$S = S^{(0)} + \sum_{n=1}^{\infty} J^n S^{(n)}$$

$$u = u^{(0)} + \sum_{n=1}^{\infty} J^n u^{(n)} \tag{50}$$

$$W = W^{(0)} + \sum_{n=1}^{\infty} J^n W^{(n)}.$$

Now we note that ψ in (46) was understood to represent the actual, i.e. perturbed, longitude of the satellite at all times. In order to make the connection between this perturbed independent variable and the time we find that

$$t - t_0 = \frac{1}{p}\int_{\psi_0}^{\psi} \frac{d\psi}{W^2} = \frac{1}{p}\int_{\psi_0}^{\psi} \frac{1}{W^{(0)2}}\left(1 - 2\,\frac{W^{(1)}}{W^{(0)}} + \cdots\right) d\psi$$
$$= t^{(0)} + \sum_{n=1}^{\infty} J^n t^{(n)} \tag{51}$$

i.e. the time itself evolves as a perturbation series. In the zero-order solution of (48) to (51) the right-hand side of (49) vanishes and we get

$$S^{(0)} = A \sin(\psi + \delta), \tag{52}$$

where A and δ are integration constants. On the right-hand side of (48) we retain the first term after substitution of $S^{(0)}$. Then $W^{(0)}$ follows in a straightforward manner. Substituting it into (51), the usual transcendental expression for time in Keplerian orbits results. It is somewhat obscured by the fact that its argument is given in terms of the longitude rather than one of the anomalies. We record the simplified form of these results for circular orbits, where $u^{(0)} = 1/r_0$:

$$W^{(0)} = \frac{1}{r_0}[1 + A^2 \sin^2(\psi + \delta)]^{\frac{1}{2}} \tag{53}$$

$$t^{(0)} = \frac{r_0^2}{2p}\left\{\frac{2}{(1 + A^2)^{\frac{1}{2}}} \tan^{-1}[(1 + A^2)^{\frac{1}{2}} \tan(\psi + \delta)]\right\}_{\psi_0}^{\psi}. \tag{54}$$

It is important to note that the expressions (52) and (53) represent Keplerian (in fact circular) motion only for the unperturbed case: i.e., if (54) represents the entire time equation and no higher-order terms as in (51) exist. For any perturbed motion, where ψ is perturbed in relation to time, the zero-order terms of course retain the Keplerian forms (52) to (54), but they do not actually represent Keplerian motion.

If we now proceed to the first-order solution and retain only terms of $0(J)$ in (48) and (49), we find

$$S^{(1)''} + S^{(1)} = -\frac{2kR^2}{p^2}\frac{S^{(0)}u^{(0)}}{[1 + S^{(0)2}]^2} \tag{55}$$

and

$$W^{(1)''} + W^{(1)} = \frac{-3kS^{(0)}S^{(1)}}{p^2[1 + S^{(0)2}]^{\frac{3}{2}}} + \frac{3kR^2u^{(0)2}}{p^2[1 + S^{(0)2}]^{\frac{3}{2}}}\left[\frac{1}{1 + S^{(0)2}} - \frac{2}{3}\right]. \tag{56}$$

Since the right-hand side of (55) contains only zero-order quantities,

we begin our solution there and the result is

$$S^{(1)} = \frac{kR^2A}{p^2r_0}\left[\frac{\sin(\psi+\delta)}{A^2[1 + A^2\sin^2(\psi+\delta)]} + \frac{\cos(\psi+\delta)}{1 + A^2}\right.$$
$$\cdot\left\{\frac{1}{\sqrt{1+A^2}}\tan^{-1}[\sqrt{1+A^2}\tan(\psi+\delta)] \right. \tag{57}$$
$$\left.\left. - \frac{\sin 2(\psi+\delta)}{1 + A^2\sin^2(\psi+\delta)}\right\}\right].$$

Here we do not show a complementary solution, since it is of the same form as $S^{(0)}$ and can be absorbed with the constants A and δ. We could now substitute (57) into (56) to find $W^{(1)}$ and then use (51) to calculate the time. However, the inverse tangent in $S^{(1)}$ constitutes a secular term, which is considered an objectionable feature for some applications.

This disadvantage can be avoided by inserting an additional transformation between ψ and the argument of $S^{(0)}$. Instead of using $\psi + \delta$ for the latter let it be

$$\sigma = \lambda\psi + \delta \tag{58}$$

where

$$\lambda = 1 + \sum_{n=1}^{\infty} J^n\lambda_n \tag{59}$$

and the λ_n are constants. This device is commonly attributed to Lindstedt.[12] To obtain the zero-order solution we need only substitute σ for the angular arguments in (52) and (53). However the equation for $S^{(1)}$ changes significantly, viz.:

$$S^{(1)\prime\prime} + S^{(1)} = \frac{-2kR^2S^{(0)}u^{(0)}}{p^2[1 + S^{(0)2}]^2} - 2\lambda_1 S^{(0)\prime\prime} \tag{60}$$

where the primes now denote differentiations with respect to σ. Thus we find

$$S^{(1)} = \lambda_1 A\sin\sigma - \frac{kR^2(A^2\cos^2\sigma - 1 - A^2)}{p^2Ar_0(1 + A^2)(1 + A^2\sin^2\sigma)}$$
$$- \cos\sigma\left\{\lambda_1 A\sigma - \frac{kR^2A}{p^2r_0(1 + A^2)^{\frac{3}{2}}}\tan^{-1}[(1 + A^2)^{\frac{1}{2}}\tan\sigma]\right\}. \tag{61}$$

The appearance of the free parameter λ_1 in this result gives us the

opportunity to suppress the secular term. Thus, if we choose

$$\lambda_1 = \frac{kR^2}{p^2 r_0 (1 + A^2)^{\frac{3}{2}}} \tag{62}$$

(61) becomes

$$S^{(1)} = \frac{kR^2 A \cos \sigma}{p^2 r_0 (1 + A^2)^{\frac{3}{2}}} \{\tan^{-1} [(1 + A^2)^{\frac{1}{2}} \tan \sigma] - \sigma\}. \tag{63}$$

Here we have again omitted all terms of the same form as $S^{(0)}$. The net contribution from the terms in braces is cyclic and has the period 2π in σ.

According to (58) and (62) this amounts to a period of

$$2\pi \left[1 - \frac{JkR^2}{r_0 p^2 (1 + A^2)^{\frac{1}{2}}} \right] \tag{64}$$

in ψ. In effect, Lindstedt's transformation distorts the independent variable to absorb the secular effect. We shall see more of this later.

In principle we could transform (56) to σ and solve for W in a straightforward manner. However, to simplify the algebra, a redefinition of W will be convenient. We may backtrack to the explicit form of (48) in terms of S, u, and σ.

$$\begin{aligned} &[1 + S^{(0)2}]^2 u^{(1)\prime\prime} + 2[1 + S^{(0)2}] S^{(0)} S^{(0)\prime} u^{(1)\prime} \\ &\quad + [S^{(0)\prime 2} + S^{(0)2} + 1] u^{(1)} \\ &\qquad = \frac{3kR^2}{p^2} u^{(0)2} \left[\frac{1}{1 + S^{(0)2}} - \frac{2}{3} \right] \\ &\qquad\quad - 2[S^{(0)} S^{(1)} + S^{(0)\prime} S^{(1)\prime} + S^{(0)\prime 2} \lambda_1] u^{(0)} \end{aligned} \tag{65}$$

and take

$$W^{(1)} = (1 + S^{(0)2})^{\frac{1}{2}} u^{(1)}. \tag{66}$$

Then we obtain*

$$\begin{aligned} W^{(1)\prime\prime} + W^{(1)} = &\frac{kR^2 u^{(0)2} [1 - 2S^{(0)2}]}{p^2 \Delta^5} \\ &- \frac{2u^{(0)}}{\Delta^3} [S^{(0)} S^{(1)} + S^{(0)\prime} S^{(1)\prime} + S^{(0)\prime 2} \lambda_1] \end{aligned} \tag{67}$$

where $\Delta = (1 + A^2 \sin^2 \sigma)^{\frac{1}{2}}$.

Substituting (63) and ignoring the complementary solution for $W^{(1)}$ we

* Note that the formulas (36) and (38) in Ref. 9 contain several misprints.

have

$$W^{(1)} = \frac{-kR^2}{3p^3r_0^2(1 + A^2)^3\Delta^3}\,[4A^4(A^4 - 1)\sin^4\sigma + A^2(9A^4 + 2A^2 - 7)\sin^2\sigma + 5A^4 + 2A^2 - 3]. \tag{68}$$

Now it only remains to find a relation between σ and time. From (46) it is clear that

$$t - t_0 = \frac{1}{p}\int_{\psi_0}^{\psi} r^2\cos^2\varphi\,d\psi = \frac{1}{p}\int_{\sigma_0}^{\sigma}\frac{d\sigma}{u^2(1 + S^2)\lambda}, \tag{69}$$

which we expand up to $0(J)$. This results in

$$t^{(0)} = \frac{r_0^2}{p(1 + A^2)^{\frac{1}{2}}}\tan^{-1}[(1 + A^2)^{\frac{1}{2}}\tan\sigma] \tag{70}$$

and

$$\begin{aligned} t^{(1)} &= -\frac{r_0^2}{p}\int_{\sigma_0}^{\sigma}\frac{1}{\Delta^2}\left[\lambda_1 + \frac{2r_0}{\Delta}W^{(1)} + \frac{2A\sin\delta\,S^{(1)}}{\Delta^2}\right]d\sigma \\ &= -\frac{kR^2r_0}{p^3(1 + A^2)^{\frac{5}{2}}}\left\{\frac{(1 + A^2)^{\frac{1}{2}}}{12\Delta^2}[A^2\sin 2\sigma + 12(1 + A^2)^{\frac{1}{2}}\sigma]\right. \\ &\quad + \left[2 - 3A^2 + \frac{A^2}{2\Delta^2}((1 + A^2)\sin^2\sigma - \cos^2\sigma)\right] \\ &\quad \left.\cdot\tan^{-1}[(1 + A^2)^{\frac{1}{2}}\tan\sigma]\right\} \end{aligned} \tag{71}$$

and completes this analysis of near-circular orbits.

Throughout the foregoing discussion we have used a perturbed coordinate, ψ or σ, as the independent variable. In principle, we could have done without Lindstedt's device, and we could have used the unperturbed longitude $\psi^{(0)}$ as the independent variable. This approach has the attractive feature that the zero-order solution (in terms of $\psi^{(0)}$) represents true Keplerian motion. The perturbed longitude could be expressed in terms of $\psi^{(0)}$ as

$$\psi = \psi^{(0)} + \sum_{n=1}^{\infty} J^n\psi^{(n)}(\psi^{(0)}). \tag{72}$$

However, if one develops the governing differential equations for $S^{(1)}$ and $W^{(1)}$, he discovers that they are completely coupled for this particular example. This approach, therefore, loses its practical value.

In conclusion we note that, in spite of various transformations, the

final results (63), (68), (71) of this analysis seem rather awkward, considering the fact that they represent the relatively trivial first-order oblateness perturbations of a near-circular orbit. This is of course due to the choice of spherical equatorial coordinates to represent the motion in a nonequatorial orbit. That disadvantage was eliminated in other formulations, to be considered next.

4.2 *Perturbations in Orbital Spherical Coordinates*

As the title of this section indicates, it is more natural to use the plane of unperturbed motion as the fundamental plane for inclined orbits. A typical analysis in this category is that by Anthony, Fosdick et al.[13,14] The coordinates r, θ, β of their reference frame (see Fig. 3) take the place of r, ψ, $(\pi/2) - \varphi$ in Fig. 2. The angle $\alpha = (\pi/2) - \varphi$ (Fig. 3) is introduced occasionally for trigonometric simplifications.

The left-hand sides of the equations of motion of course are not altered by this change of coordinates, but the right-hand sides (representing oblateness perturbations) acquire the forms shown in (73) to (75). As usual, we introduce $u = 1/r$ and $p = r^2\dot{\theta} = \dot{\theta}/u^2$ and change the independent variable from t to θ. We note that θ is the perturbed central angle in the nominal orbit plane. Thus, the equations of motion become

$$(pu') + pu(\beta'^2 + \sin^2 \beta) = (k/p)[1 + JR^2u^2(1 - 3\cos^2 \alpha)] \tag{73}$$

$$\begin{aligned}(p\theta')' - p \sin \beta \cos \beta = (-kJR^2u/p)[(\sin^2 i \sin^2 \theta \\ - \cos^2 i) \sin 2\beta \\ + \sin 2i \cos 2\beta \sin \theta]\end{aligned} \tag{74}$$

$$\begin{aligned}(p \sin^2 \beta)' = (-kJR^2u/p)[\sin^2 i \sin^2 \beta \sin 2\theta \\ + \tfrac{1}{2} \sin 2i \sin 2\beta \cos \theta]\end{aligned} \tag{75}$$

where primes denote derivatives with respect to θ. We subject this variable to the first-order Lindstedt transformation

$$\sigma = \theta(1 + J\lambda_1) \tag{76}$$

and use the "ansatz"

$$\begin{aligned} u &= u^{(0)}(\sigma) + Ju^{(1)}(\sigma) \\ p &= p^{(0)}(\sigma) + Jp^{(1)}(\sigma) \\ \beta &= (\pi/2) + J\beta^{(1)}(\sigma). \end{aligned} \tag{77}$$

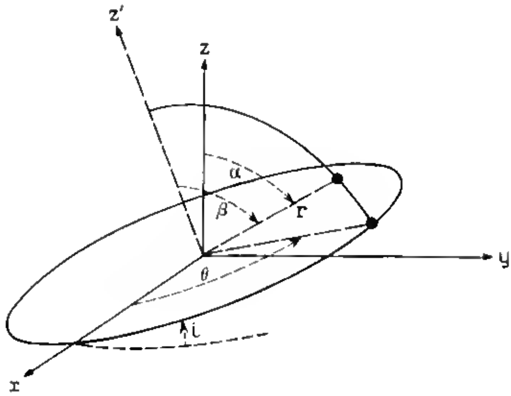

FIG. 3 — Orbital spherical coordinate system.

Without an excessive loss of generality we may assume a horizontal launch at the node and, by the definition of the θ plane as nominal orbit plane, the initial velocity vector $\mathbf{v}_0$ lies in it. Thus, at $t = 0$: $r = r_0$, $\dot{r} = 0$, $\beta = \pi/2$, $\dot{\beta} = 0$, and $\dot{\theta} = v_0/r_0$. In general, v_0 will be such as to produce an elliptic orbit. The zero-order results are

$$p^{(0)} = r_0 v_0$$

and

$$u^{(0)} = (k/r_0^2 v_0^2)[1 + e \cos \sigma] \tag{78}$$

where

$$e = (r_0 v_0^2/k) - 1$$

and has the form of a Keplerian eccentricity. As in Section 4.1, the zero-order solution (78) will represent Keplerian motion only for the unperturbed case, i.e. when $\sigma = \theta = \theta^{(0)}$. Now the first-order solutions follow in a straightforward manner:

$$p^{(1)} = (-k^2/2r_0 v_0^3)(R/r_0)^2 \sin^2 i \{1 + \tfrac{4}{3}e - e \cos \sigma - \tfrac{1}{3}e \cos 3\sigma - \cos 2\sigma\} \tag{79}$$

$$\begin{aligned} u^{(1)} = (k^3 R^2/r_0^6 v_0^6)\{1 &+ \tfrac{1}{2}e^2 + \sin^2 i(-\tfrac{1}{2} + \tfrac{4}{3}e - \tfrac{5}{8}e^2) \\ &- \tfrac{1}{6}(e^2 + \sin^2 i) \cos 2\sigma - \tfrac{5}{24}e \sin^2 i \cos 3\sigma \\ &- \tfrac{1}{24}e^2 \sin^2 i \cos 4\sigma \\ &- [1 + \tfrac{1}{3}e^2 - \sin^2 i(\tfrac{2}{3} - \tfrac{9}{8}e + \tfrac{2}{3}e^2)] \cos \sigma\}, \end{aligned} \tag{80}$$

where we had to select the Lindstedt parameter as

$$\lambda_1 = (k^2R^2/r_0^4v_0^4)(\tfrac{3}{2}\sin^2 i - 1) \tag{81}$$

to avoid a secular term in (80). Finally

$$\beta^{(1)} = \frac{k^2R^2 \sin 2i}{2r_0^4v_0^4}[(1 + \tfrac{2}{3}e)\sin\sigma - \sigma\cos\sigma - \tfrac{1}{3}e\sin 2\sigma]. \tag{82}$$

It is clear that the secular term which was absorbed by the Lindstedt parameter has to do with the apsidal precession. In the absence of additional Lindstedt parameters we have no countermeasures against the secular term in (82), which reflects the nodal regression. (Note that the latter was counteracted by the Lindstedt transformation of Section 4.1, since it was the only secular effect to be considered for near-circular orbits.)

The time equation for this example can be written in a straightforward fashion. From the definition of p it follows that

$$t_1 - t_0 = \int_{\sigma_0}^{\sigma_1} \frac{(1 - J\lambda_1)}{pu^2}\, d\sigma$$

where σ_0 and σ_1 correspond to the time limits t_0 and t_1. An expansion to first-order terms yields

$$\begin{aligned} t^{(0)} + Jt^{(1)} = &\int_{\sigma_0}^{\sigma_1} \frac{d\sigma}{p^{(0)}u^{(0)2}} \\ &- J\int_{\sigma_0}^{\sigma_1} \frac{1}{p^{(0)}u^{(0)2}}\left[\lambda_1 + \frac{p^{(1)}}{p^{(0)}} + 2\frac{u^{(1)}}{u^{(0)}}\right] d\sigma. \end{aligned} \tag{83}$$

This leads to

$$\begin{aligned} t^{(0)} = \frac{r_0^3v_0^3}{k^2}\Bigg\{&\frac{-e\sin\sigma}{(1 - e^2)(1 + e\cos\sigma)} \\ &+ \frac{2}{(1 - e^2)^{\frac{3}{2}}}\tan^{-1}\left[\left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}}\tan\frac{\sigma}{2}\right]\Bigg\}_{\sigma_0}^{\sigma_1}, \end{aligned} \tag{84}$$

which we recognize as being of strictly Keplerian form but in terms of the perturbed angle σ. Similarly

$$
\begin{aligned}
t^{(1)} = \frac{R^2}{r_0 v_0} \Bigg\{ & \frac{\sin \sigma (4 - e^2 + 3e \cos \sigma)}{(1 - e^2)^2 (1 + e \cos \sigma)^2} \Bigg[1 + \tfrac{2}{3} e + \tfrac{1}{3} e^2 + \tfrac{2}{3} e^3 \\
& + \left(\frac{1}{6e} - \tfrac{2}{3} + \tfrac{1}{2} e + \tfrac{2}{3} e^2 - \tfrac{2}{3} e^3 \right) \sin^2 i \Bigg] \\
& - \frac{\sin \sigma}{(1 - e^2)(1 + e \cos \sigma)} \Bigg[2 - \tfrac{1}{3} e + \tfrac{2}{3} e^2 \\
& + \left(\frac{2}{3e} - \tfrac{4}{3} + \tfrac{11}{6} e - \tfrac{2}{3} e^2 \right) \sin^2 i \Bigg] \\
& - \frac{[2(1 + e)^3 + \frac{4}{3} e^3 \sin^2 i]}{(1 - e^2)^{\frac{5}{2}}} \tan^{-1} \left[\left(\frac{1 - e}{1 + e} \right)^{\frac{1}{2}} \tan \frac{\sigma}{2} \right] \Bigg\}_{\sigma_0}^{\sigma_1}.
\end{aligned}
\tag{85}
$$

The set of results (78) to (85) gives a reasonably convenient description of first-order oblateness perturbations which might be useful in the targeting and guidance of space vehicles. Extensions to near-parabolic and hyperbolic trajectories follow quite readily. As in Section 4.1, we note that the analysis might have been executed in terms of an unperturbed independent variable, viz. $\theta^{(0)}$ instead of θ, and in that case the zero-order solution would represent true Keplerian motion.

The inclusion of secular perturbations in the independent variable σ serves the same purpose as the definitions of "mean elements" introduced by Breakwell et al., by Hansen, in the von Zeipel method, and in modern averaging techniques. The Lindstedt transformation is not the most powerful device in this category but it can be extended to absorb secular effects in more than one coordinate. This will be illustrated in the next section in terms of "secular rotations" of the reference frame.

4.3 *Perturbations in Rotating Spherical Orbital Coordinates*

The idea of using suitable coordinate transformations with arbitrary parameters to neutralize secular trends was exploited in a more general way by R. E. Roberson.[15] His approach uses the orbital coordinates r, θ, δ $[= (\pi/2) - \beta]$, in agreement with Section 4.2, but assumes that the entire reference frame will be subjected to three monotonic rotations, corresponding to three Euler angles, such that the satellite motion relative to this reference frame exhibits only periodic perturbations. This kinematic outlook on secular trends forms an interesting parallel to several classical procedures. Roberson himself makes some illuminating comparisons between engineering analyses such as Refs. 9, 10, and 12 to 16 and traditional formulations in terms of mean variables. He restricts his analysis to first-order perturbations, realizing that a con-

sistent higher-order theory would have to include contributions from other physical effects and various coupling terms. Some of his remarks seem quite perspicacious in comparison with the other aerospace literature of that time.

The angular velocities stipulated for the reference frame must of course depend on the secular effects that need to be absorbed. In the presence of several physical disturbances the different angular motions of the coordinate system can be superposed to first order, and the resultant motion of the reference frame will succeed in neutralizing all the secular effects simultaneously. This seems intuitively obvious and can be demonstrated in a straightforward fashion.[15]

In Fig. 4 the angles $\bar{\Omega}$ and $\bar{\imath}$ define a mean orbit plane, in that each of them manifests a secular rate. Now the satellite position is given in terms of the orthogonal system $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, which displays a secular variation with respect to the node (and this corresponds to the third Eulerian rotation). Let the three secular rotations be denoted $\kappa\dot{\Omega}^{(1)}$, $\kappa(di^{(1)}/dt)$, $\kappa\dot{\omega}^{(1)}$ where κ is the perturbation parameter. They will in general be functions of $\bar{\Omega}$, $\bar{\imath}$ and the characteristics of the perturbation source.

Turning to the problem of first-order oblateness perturbations, we set $\kappa = J$ and assume the appropriate form for the perturbing potential. Now let

$$\theta = \theta_0 + \bar{f} = \theta_0 + \bar{f}^{(0)} + J\bar{f}^{(1)}, \tag{86}$$

where $\bar{f} = 0$ at $t = 0$. We adopt $\bar{f}$ as independent variable and let de-

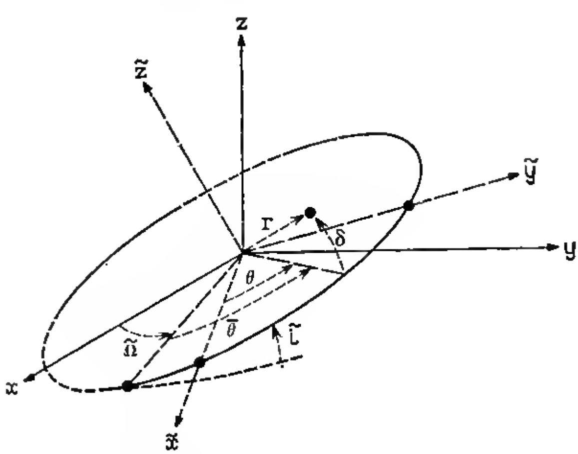

Fig. 4 — Rotating spherical orbital coordinates.

rivatives with respect to it be denoted by primes. Assuming that the secular rates are constants, we let

$$\begin{aligned}\bar{\Omega} &= \Omega^{(0)} + J\hat{\Omega}^{(1)}\bar{f}\\ \bar{i} &= i^{(0)} + J\hat{i}^{(1)}\bar{f}\\ \bar{\theta} &= \theta + J\hat{\omega}^{(1)}\bar{f}.\end{aligned} \tag{87}$$

As usual, the equations of motion are transformed by means of

$$u = 1/r \quad \text{and} \quad p = r^2\dot{\bar{f}} \tag{88}$$

and we find:

$$\begin{aligned}(pu')' + pu[(\delta' - \hat{\Omega}^{(1)} \cos\theta \sin i)^2\\ + (\cos\delta + \hat{\omega}^{(1)}\cos\delta + \hat{\Omega}^{(1)}\cos\varphi\cos\nu)^2]\\ - (k/p)[1 + 3JR^2u^2(1 - 3\sin^2\varphi)] = 0\end{aligned} \tag{89}$$

$$\begin{aligned}[p(\delta' - \hat{\Omega}^{(1)} \sin i \cos\theta)]' + p[(1 + \hat{\omega}^{(1)})^2 \sin\delta\cos\delta\\ + (1 + \hat{\omega}^{(1)})\hat{\Omega}^{(1)}(\sin\delta\cos\nu\cos\varphi + \cos\delta\sin\varphi)\\ + \hat{\Omega}^{(1)2}\sin\varphi\cos\nu\cos\varphi] + (JkR^2 6u/p)\sin\varphi\cos\varphi\cos\nu = 0\end{aligned} \tag{90}$$

$$\begin{aligned}[p\cos\delta\,(\bar{\theta}'\cos\delta + \hat{\Omega}^{(1)}\cos\varphi\cos\nu)]'\\ + p[(1 + \hat{\omega}^{(1)})\hat{\Omega}^{(1)}\sin\delta\cos\delta\sin i\cos\theta\\ + \hat{\Omega}^{(1)2}\cos\theta\sin i\sin\delta\cos\nu\cos\varphi\\ - \delta'\hat{\Omega}^{(1)}\sin i\sin\theta]\\ + (6JkR^2u/p)\sin\varphi\cos\delta\sin i\cos\theta = 0,\end{aligned} \tag{91}$$

where φ is the latitude and ν is defined in Fig. 5. We have made a slight digression from orderly progress in this step by setting $\hat{i}^{(1)} = 0$. This is prompted by previous experience with this problem — viz., that no first-order secular perturbations occur in i — and would have developed from the later calculations in any event.

Using the forms

$$\begin{aligned}u &= u^{(0)} + Ju^{(1)}\\ p &= p^{(0)} + Jp^{(1)}\\ \delta &= J\delta^{(1)}\end{aligned} \tag{92}$$

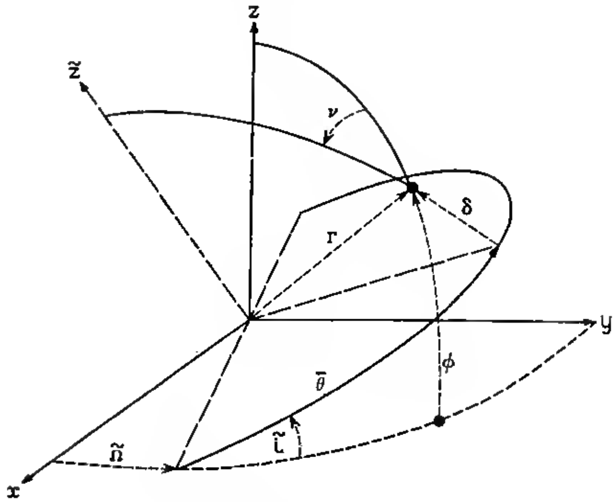

Fig. 5 — Definition of ν.

we reduce (89)–(91) to equations of $0(1)$ and $0(J)$. The zero-order solution is of course

$$\delta^{(0)} \equiv 0, \qquad p^{(0)} = \text{const.}, \qquad \text{and} \qquad u^{(0)} = (k/p^{(0)2})[1 + e\cos(\bar{f} - \alpha)]. \tag{93}$$

As in previous examples, we see that this will represent Keplerian motion only in the absence of perturbations, i.e. if $\bar{f} \equiv \bar{f}^{(0)}$ and $\hat{\Omega}^{(1)} = \hat{\omega}^{(1)} = 0$, yielding an inertial reference frame. We assume that the constants of integration $(p^{(0)}, e, \alpha)$ are chosen such that (93) with $\bar{f} = 0$ yields the satellite position and velocity at $t = 0$.

Proceeding with the solutions to $0(J)$ in the usual fashion, we require that

$$\hat{\Omega}^{(1)} = -\frac{3Jk^2R^2\cos i^{(0)}}{p^{(0)4}} \tag{94}$$

in order to avoid a secular term in $\delta^{(1)}$ and

$$\hat{\omega}^{(1)} = (3R^2k^2/2p^{(0)4})(5\cos^2 i^{(0)} - 1) \tag{95}$$

to avoid one in $u^{(1)}$. These of course reflect the nodal and apsidal precessions. The complementary solution for $p^{(1)}$ introduces one constant of integration, and the complementary solutions for $\delta^{(1)}$ and $u^{(1)}$ have

the form

$$A \sin \bar{f} + B \cos \bar{f}. \tag{96}$$

Since the zero-order solution already accounts for the dynamic state of the satellite at $t = 0$, the first-order solution encounters homogeneous initial conditions as far as they do not reflect the rotation of the reference frame. Thus at $\bar{f} = 0$:

$$u^{(1)} = \delta^{(1)} = 0, \qquad u^{(1)\prime} = 0$$

$$\delta^{(1)\prime} - \hat{\Omega}^{(1)} \sin i^{(0)} \cos \theta_0 = 0,$$

and

$$(1/\dot{\bar{f}}^{(0)})(2u^{(1)}u^{(0)}p^{(0)} + u^{(0)2}p^{(1)}) + \hat{\omega}^{(1)} + \hat{\Omega}^{(1)}(\cos i^{(0)} - \sin i^{(0)} \sin \theta_0) = 0. \tag{97}$$

These govern the first-order constants of integration. [A little reflection shows that the forms (96) for $\delta^{(1)}$ and $u^{(1)}$ can be interpreted geometrically as constant changes of Ω and θ to $0(J)$. Roberson anticipates this by introducing such constants in (87) and using them in place of two of the integration constants for $\delta^{(1)}$ and $u^{(1)}$. However, nothing seems to be gained by this artifice and, if anything, it distracts from a systematic procedure.]

Finally, the time equation follows as usual in terms of $\bar{f}$ to $0(J)$. Roberson proceeds to invert it, though the computational gains do not seem to justify this algebraic labor.

So much for our sketch of Roberson's procedure. Its extension to higher-order analyses is fairly obvious. At every level of refinement, $0(J^n)$, three coordinate rotations may be introduced — which are commensurate with the three degrees of freedom of the satellite problem whose secular trends we are trying to neutralize.

For "medium-range" prediction formulas it seems an open issue whether the rationale described in this section and traditional astronomical devices (like the "auxiliary ellipse" used by Hansen or the von Zeipel transformations based on Hamilton-Jacobi techniques) offer a computational advantage over the straightforward development of the Poisson method for successive higher-order terms. With the advent of computer algebra the latter technique may be quite satisfactory for many applications. However, for "long-range" predictions and lifetime studies it seems advisable to employ the accredited astronomical techniques of "extracting" secular effects and anticipating long-period terms in one way or another.

V. MOVING RECTANGULAR ORBITAL COORDINATES

We close this article with a formulation which calculates the position offsets for a satellite from its unperturbed orbit in an explicit form.[18,19] Instead of reckoning the perturbations in terms of the quantities r, θ, β or δ, which are defined relative to the center of the earth, we now consider a coordinate system whose origin is the nominal satellite position $0'$ on the unperturbed orbit (see Fig. 6). The latter may be defined by the initial conditions at time t_0, viz. $\mathbf{r}_0$ and $\mathbf{v}_0$. We establish an orthogonal triad about the moving point $0'$ with ξ in the radial direction, η in the direction of anomalistic motion, and ζ normal to the orbit plane. In the guidance engineer's language

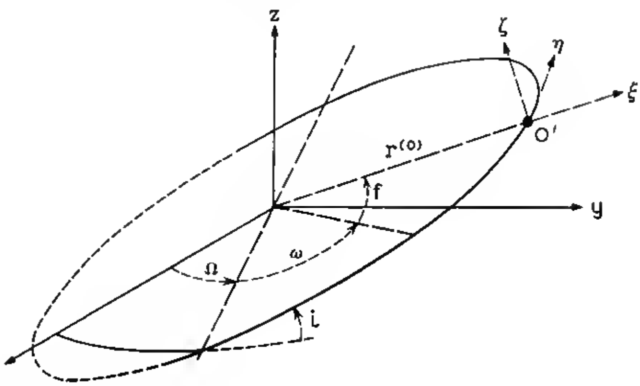

FIG. 6 — Moving rectangular coordinates centered at nominal satellite position.

these represent offsets in altitude, range, and cross-range. Any nonvanishing coordinates in this system are the effects of errors in the initial conditions or of geophysical forces. It is clear that this description of the perturbed motion can be quite useful in guidance studies, e.g. to exhibit the relative motion between a space station (given by $0'$) and a transfer vehicle (located at ξ, η, ζ) in a homing maneuver. In the subsequent discussion f will always represent the unperturbed true anomaly in the nominal orbit and $\theta = \omega + f$ the unperturbed central angle.* No Lindstedt transformations or perturbative coordinate rotations will be employed to develop this theory into a more sophisticated prediction scheme. Instead, we concentrate on the geometric interpretation of various results.

* We depart from earlier notations by omitting the superscript (0) from unperturbed quantities for simplicity.

The equations of motion can be derived by the standard Lagrangian or Hamiltonian formalism,[18] and their linearization to $0(\xi, \eta, \zeta)$ yields

$$\xi'' - 2\eta' - \xi - 2[\xi + e(\xi' - \eta)\sin f]/(1 + e\cos f) = -\frac{a^3(1-e^2)^3}{k}\tilde{V}_\xi/(1 + e\cos f)^4 \tag{98}$$

$$\eta'' + 2\xi' - \eta - [-\eta + 2e(\eta' + \xi)\sin f]/(1 + e\cos f) = -\frac{a^3(1-e^2)^3}{k}\tilde{V}_\eta/(1 + e\cos f)^4 \tag{99}$$

$$\zeta'' + [\zeta - 2\zeta' e\sin f]/(1 + e\cos f) = -\frac{a^3(1-e^2)^3}{k}\tilde{V}_\zeta/(1 + e\cos f)^4 \tag{100}$$

where primes denote derivatives with respect to f and $\tilde{V}$ is the perturbative potential, which exists in addition to the central body term $-k/r$. The subscripts of $\tilde{V}$ denote partial derivatives with respect to ξ, η, or ζ.

The solution of the homogeneous set (98) and (99), where $\tilde{V} \equiv 0$, represents a complementary solution for the cases where $\tilde{V} \not\equiv 0$ and will be needed to satisfy the initial conditions. For an elliptic nominal orbit this solution has the form

$$\xi = \delta a\left[\frac{1-e^2}{1+e\cos f} - \frac{3e}{2}\left(\frac{k}{(1-e^2)a^3}\right)^{\frac{1}{2}}(t-\tau)\sin f\right] - \delta e\, a\cos f - \delta\tau\, e\left(\frac{k}{(1-e^2)a}\right)^{\frac{1}{2}}\sin f \tag{101}$$

$$\eta = -\delta a\frac{3}{2}\left(\frac{k}{(1-e^2)a^3}\right)^{\frac{1}{2}}(1+e\cos f)(t-\tau) + \delta e\, a\sin f\,\frac{(2+e\cos f)}{1+e\cos f} - \delta\tau\left(\frac{k}{(1-e^2)a}\right)^{\frac{1}{2}} \cdot (1+e\cos f) + \delta\omega\,\frac{a(1-e^2)}{1+e\cos f} \tag{102}$$

$$\zeta = \frac{a(1-e^2)}{(1+e\cos f)}[\delta i\sin\theta - \delta\Omega\sin i\cos\theta]. \tag{103}$$

This result can be adapted to hyperbolic, parabolic, and near-parabolic orbits without much trouble. The constants of integration $\delta a \cdots \delta\tau$ are of course just a set of numbers to be determined from the initial conditions, but the symbols we use for them indicate the parameter changes

of the nominal orbit that they represent. Alternatively, these constants could be given in terms of $\xi_0 \cdots \zeta_0'$, the perturbations of the position and velocity at t_0. That form is more descriptive for various guidance applications. Thus we find for nominally circular orbits

$$\xi = 2\eta_0' + 4\xi_0 - (2\eta_0' + 3\xi_0) \cos (f - f_0) + \xi_0' \sin (f - f_0) \tag{104}$$

$$\begin{aligned} \eta = \eta_0 - 2\xi_0' - 3(\eta_0' + 2\xi_0)(f - f_0) \\ + 2(2\eta_0' + 3\xi_0) \sin(f - f_0) + 2\xi_0' \cos (f - f_0) \end{aligned} \tag{105}$$

$$\zeta = \zeta_0' \sin (f - f_0) + \zeta_0 \cos (f - f_0). \tag{106}$$

(Since a nominal perigee does not exist for this case, we assume that the angle ω, whose existence is still implied by the notation f, has some arbitrary value $\bar{\omega}$. Without loss of generality we can set $f_0 = 0$ so that $\theta_0 = \bar{\omega}$.)

In a guidance application $\xi_0 \cdots \zeta_0'$ might represent the errors resulting from a position and velocity determination or a corrective thrust maneuver and (104) to (106) would then describe the "run-out" as a function of time. In an "orbit sensitivity" study these expressions can be used to demonstrate the effect of $\xi_0 \cdots \zeta_0'$ on the orbit parameters. In a homing maneuver the same expressions would represent the relative motion between the two vehicles attempting a rendezvous. In principle, two relative position measurements $\mathbf{X}$: ξ, η, ζ at separate times suffice to determine all the constants in (104) to (106), and a corrective maneuver could be planned to drive the residuals to zero at a specified instant or by successive approximations.

Particular solutions of (98) to (100) can be found in a straightforward manner if $e = 0$. For $e \neq 0$ we construct these solutions as power series in e, for lack of a better expedient. We consider the series to $0(e)$ and let them be denoted by

$$\begin{aligned} \xi &= \bar{\xi}_1 + \bar{\xi}_2 + e \sum_j f_j \\ \eta &= \bar{\eta}_1 + \bar{\eta}_2 + e \sum_j g_j \qquad j = 1, 2, 3 \\ \zeta &= \bar{\zeta}_1 + \bar{\zeta}_2 + e \sum_j h_j \end{aligned} \tag{107}$$

where

$\bar{\xi}_1\bar{\eta}_1\bar{\zeta}_1 =$ the complementary solution (104) to (106)
$\bar{\xi}_2\bar{\eta}_2\bar{\zeta}_2 =$ a particular solution representing $\tilde{V}$ to $0(\kappa)$
$f_1 g_1 h_1 =$ solution reflecting $e \cdot (\bar{\xi}_1, \bar{\eta}_1, \bar{\zeta}_1)$ on the right-hand sides of (98) to (100)

$f_2g_2h_2$ = solution reflecting $e\cdot(\bar{\xi}_2, \bar{\eta}_2, \bar{\zeta}_2)$ on the right-hand sides
$f_3g_3h_3$ = solution reflecting $e\cdot(\tilde{V}_\xi, \tilde{V}_\eta, \tilde{V}_\zeta)$ on the right-hand sides.

The following explicit general forms can be given for these solutions:

$$\bar{\xi}_2 = (a^3/k)\Big[-2\int \tilde{V}_\eta\,df + 2\cos f\int \tilde{V}_\eta \cos f\,df$$
$$+ \cos f\int \tilde{V}_\xi \sin f\,df + 2\sin f\int \tilde{V}_\eta \sin f\,df$$
$$- \sin f\int \tilde{V}_\xi \cos f\,df\Big]$$
$$\bar{\eta}_2 = (a^3/k)\Big[3\iint \tilde{V}_\eta\,df\,df + 2\int \tilde{V}_\xi\,df - 4\sin f\int \tilde{V}_\eta \cos f\,df \qquad (108)$$
$$- 2\sin f\int \tilde{V}_\xi \sin f\,df + 4\cos f\int \tilde{V}_\eta \sin f\,df$$
$$- 2\cos f\int \tilde{V}_\xi \cos f\,df\Big]$$
$$\bar{\zeta}_2 = (a^3/k)\Big[\cos f\int \tilde{V}_\zeta \sin f\,df - \sin f\int \tilde{V}_\zeta \cos f\,df\Big]$$

where we take $f_0 = 0$ for the lower limit of all quadratures.

$$f_1 = (\eta_0 - 2\xi_0')\sin f - \tfrac{5}{2}(\eta_0' + 2\xi_0)\cos f - 3(\eta_0' + 2\xi_0)$$
$$\cdot f \sin f$$
$$g_1 = 7(\eta_0' + 2\xi_0)\sin f + (\eta_0 - 2\xi_0')\cos f - 3(\eta_0' + 2\xi_0) \qquad (109)$$
$$\cdot f\cos f - (\xi_0'/2)\cos 2f - (\eta_0' + \tfrac{3}{2}\xi_0)\sin 2f$$
$$h_1 = -(\zeta_0/2) - (\zeta_0'/2)\sin 2f - (\zeta_0/2)\cos 2f.$$

The terms f_2, g_2, and h_2 are obtainable from expressions analogous to (108) but with the following substitutions:

$$2(\bar{\xi}_2' - \bar{\eta}_2)\sin f - 2\bar{\xi}_2\cos f \qquad \text{for } (-a^2/k)\tilde{V}_\xi$$
$$2(\bar{\eta}_2' + \bar{\xi}_2)\sin f + \bar{\eta}_2\cos f \qquad \text{for } (-a^3/k)\tilde{V}_\eta \qquad (110)$$
$$2\bar{\zeta}_2'\sin f + \bar{\zeta}_2\cos f \qquad \text{for } (-a^3/k)\tilde{V}_\zeta$$

and $f_3g_3h_3$ follow from (108) if we substitute

$$(4a^3/k)\tilde{V}_{\xi,\eta,\zeta} \qquad \text{for } (-a^3/k)\tilde{V}_{\xi,\eta,\zeta}\,. \qquad (111)$$

Since the differential operators for all of these partial solutions are of the form

$$
\begin{aligned}
&\xi'' - 2\eta' - 3\xi \\
&\eta'' + 2\xi' \\
&\zeta'' + \zeta,
\end{aligned} \tag{112}
$$

no explicit complementary solution of $0(e)$ is provided: i.e., the constants $\xi_0 \cdots \zeta_0'$ will be used to satisfy the i.c.'s to all levels of accuracy. These will differ from zero only if the nominal orbit is taken to differ from $\mathbf{r}_0$ and $\mathbf{v}_0$ at t_0.

Specific results may now be obtained by the above formulas, which lend themselves to a geometric interpretation of perturbed satellite motions. For the oblateness effect one finds

$$
\begin{aligned}
\bar{\xi}_2 &= (JR^2/a)[-1 + \sin^2 i(\tfrac{3}{2} + \tfrac{1}{6}\cos 2\theta)] \\
\bar{\eta}_2 &= (JR^2/a)[(2 - 3\sin^2 i)f + \tfrac{1}{12}\sin^2 i \sin 2\theta] \\
\bar{\zeta}_2 &= (JR^2/2a)\sin 2i[f\cos\theta - \tfrac{1}{2}\sin\theta].
\end{aligned} \tag{113}
$$

The terms in 2θ reflect the doubly symmetric distortion of the orbit due to the oblateness of the gravitational field. The constant term in $\bar{\xi}_2$ and the secular term in $\bar{\eta}_2$ reflect the additional mass of the equatorial bulge. Combining (113) with (104) to (106) into a complete solution, we observe that the constant term in ξ is

$$\Delta a = (JR^2/a)[-1 + \tfrac{3}{2}\sin^2 i] + 2\eta_0' + 4\xi_0$$

and the secular term in η (114)

$$a\Delta\theta = f[(JR^2/a)(2 - 3\sin^2 i) - 3(\eta_0' + 2\xi_0)],$$

which represent the differences between the nominal circular orbit and the mean circular orbit resulting from the perturbations. Since $\Delta a = 0$ and $\Delta\theta = 0$ do not yield linearly independent conditions for ξ_0 and η_0', we cannot effect a launch so that the radius and the mean angular rate coincide with the nominal ones (determined for a spherical earth) unless $\sin i = \sqrt{2/3}$. On the other hand, it turns out that we can preserve the nominal inclination of the orbit by choosing $\zeta_0 = 0$ and

$$\zeta_0' = (JR^2/2a)\sin 2i\cos\theta_0. \tag{115}$$

Now, if we designate $\Delta\bar{\zeta}_2 = [\bar{\zeta}_2]_{\theta=0}^{\theta=2\pi}$, we find for the nodal regression

$$\dot{\Omega} = \frac{-\Delta\bar{\zeta}_2 k^{\frac{1}{2}}}{\sin i\, 2\pi a^{\frac{3}{2}}} = -nJ\left(\frac{R}{a}\right)^2 \cos i \tag{116}$$

and this agrees with the well-known result.

In the case of drag perturbations one replaces $\tilde{V}_\xi$, $\tilde{V}_\eta$, $\tilde{V}_\zeta$ of (110) by the appropriate components of (11):

$$
\begin{aligned}
F_\xi &= 0 \\
F_\eta &= -(C_D A\rho_0/2m)[(k/a)^{\frac{1}{2}} - \sigma a \cos i](k/a)^{\frac{1}{2}} \qquad (117)\\
F_\zeta &= -(C_D A\rho_0/2m)\,\sigma \sin i (ka)^{\frac{1}{2}} \cos\theta
\end{aligned}
$$

and finds

$$
\begin{aligned}
\bar{\xi}_2 &= (2a^3/k)F_\eta f \\
\bar{\eta}_2 &= (a^3F_\eta/k)[4 - \tfrac{3}{2}f^2] \qquad (118)\\
\bar{\zeta}_2 &= (a^3F_\zeta/4k)[2f \tan\theta - \cos\theta].
\end{aligned}
$$

Noting that

$$
\frac{1}{2\pi a}[\bar{\zeta}_2']_{\theta=0}^{\theta=2\pi} = \frac{di}{df} = -\frac{C_D A\rho_0 a^{\frac{3}{2}}\sigma \sin i}{4mk^{\frac{1}{2}}}, \qquad (119)
$$

we have agreement with standard results for the orbital precession due to diurnal winds.

If we extend this drag analysis to $0(e)$ we find

$$
\begin{aligned}
f_2 + f_3 &= (a^3F_\eta/2k)[\tfrac{5}{2}\sin f - f\cos f - 3f^2 \sin f] \\
g_2 + g_3 &= (a^3F_\eta/2k)[6f \sin f + 9\cos f - 3f^2 \cos f] \qquad (120)\\
h_2 + h_3 &= [a^3F_\zeta/(4k\cos\theta)][\tfrac{3}{2}\cos(\bar{\omega} + 2f) - \tfrac{11}{2}\cos\bar{\omega} - f\sin\bar{\omega} \\
&\quad - f\sin(\bar{\omega} + 2f)],
\end{aligned}
$$

which are simple enough to permit a further extension to cases where $\rho_0 = \rho(f)$ is variable around the orbit. The details are straightforward.[19]

It is of course understood that any of these results should be accompanied by $\bar{\xi}_1\bar{\eta}_1\bar{\zeta}_1$ if a general solution is desired. This, however, adds nothing to the characteristics of a particular perturbation. The formulas (104) to (106) and (108) to (111) can also be applied to a variety of other effects such as luni-solar perturbations and radiation pressure.

The motivation behind the results of this section was to give a geometrically tangible account of perturbed satellite motions over a fractional period or just a few periods. This may be useful in various targeting, intercept, and rendezvous operations. On the other hand, the formulations of Sections III, 4.2, and 4.3 form the beginnings of ephemeris computing techniques and orbit lifetime studies. These subjects

have been pursued further in several higher-level methods (see Section I), some of which deal partly with the elements and partly with coordinates and make occasional use of contact transformations. They may be considered a stepping stone to full-fledged astronomical perturbation analyses, about which there exists an extensive literature.

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